Greedy Column Subset Selection: New Bounds and Distributed Algorithms

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ICML 2016

Google

Talk Outline

1. Background/motivation for Column Subset Selection (CSS)
2. Previous work **+** our contributions
3. (Single-machine) greedy algorithm
4. (Distributed) coreset greedy algorithm
5. Further optimizations
6. Experiments
7. [Time permitting] Proof sketches

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Low-Rank Approximation

Given (large) matrix A in $R^{m \times n}$ and target rank k << m,n:

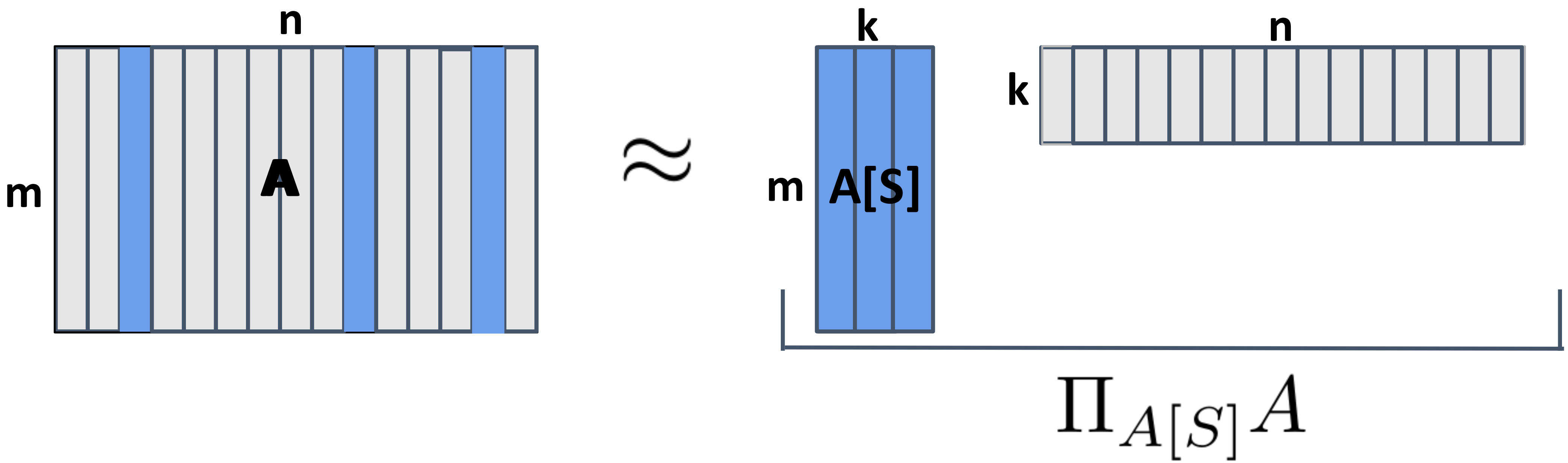
$$\underset{X,\ \mathrm{rank}(X)=k}{\arg\min} \quad \|A - X\|_F^2$$

- Optimal solution: k-rank SVD
- Applications:
  - Dimensionality reduction
  - Signal denoising
  - Compression
  - …

Column Subset Selection (CSS)

- Columns often have important meaning
- **CSS:** Low-rank matrix approximation in column space of A

$$\underset{S \subset [n],\ |S| = k}{\arg\min} \ \|A - \Pi_{A[S]} A\|_F^2$$



Why use CSS for dimensionality reduction?

- **Unsupervised**
  - Don't need labeled data
- **Classifier independent**
  - Can reuse output for different classifiers
- **Interpretable**
  - Generate features by subselecting instead of arbitrary function
- **Efficient during inference**
  - Feature subselection (CSS) better than matrix multiplication (SVD) if:
    - **Latency sensitive**
    - **SVD projection matrix prohibitively large**
    - **Sparse**

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(Very simplified) background on CSS

- CSS is **UG-hard** [Civril 2014]

- **Importance sampling** [Drineas et al. 2004, Frieze et al. 2004, …]
  - Fast, but additive-error bounds

- **More complicated algorithms** [Desphande et al. 2006, Drineas et al. 2006, Boutsidis et al. 2009, Boutsidis et al. 2011, Cohen et al. 2015, …]
  - Multiplicative-error bounds, but complicated → not as fast/distributable

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- **Greedy** [Farahat et al. 2011, Civril et al. 2011, Boutsidis et al. 2015]
  - Multiplicative-error bounds and fast/distributable

Contributions

- Prove **tight approximation guarantee** for the greedy algorithm
- **First distributed implementation with provable approximation factors**
- **Further optimizations** for the greedy algorithm
- **Empirical results** showing these algorithms are **extremely scalable** and have **accuracy comparable with the state-of-the-art**

Generalized Column Subset Selection (GCSS)

CSS (A, k) $$\arg\min_{S\subseteq[n],\,|S|=k} \|A - \Pi_{A[S]}A\|_F^2$$

GCSS(A, B, k) $$\arg\min_{S\subseteq[n],\,|S|=k} \|A - \Pi_{B[S]}A\|_F^2$$

- GCSS(A, B, k) uses k columns of B to approximate A
- Note: GCSS(A, A, k) = CSS(A, k)

Convenient reformulation of GCSS

$$\underset{S\subset[n],\,|S|=k}{\arg\max}\ \|\Pi_{B[S]}A\|_F^2 = \underset{S\subset[n],\,|S|=k}{\arg\min}\ \|A - \Pi_{B[S]}A\|_F^2$$

denote by f(S)

original GCSS cost function

- GCSS $\Longleftrightarrow$ **maximizing f subject to cardinality constraint**
- Intuition: f measures how much of A is "covered/explained" by selected columns

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GREEDY algorithm to maximize f

$S \leftarrow \varnothing$

for $i = 1 : k$

$\quad$ Pick column B_j that maximizes $f(S \cup \{B_j\})$

$\quad$ $S \leftarrow S \cup \{B_j\}$

Return S

Our result: Analysis of GREEDY

Consider $\mathrm{GCSS}(A, B, k)$ with accuracy parameter $\varepsilon > 0$. Let OPT_k be the optimal set of k columns from B. If $r = O\left(\frac{k}{\varepsilon\,\sigma_{\min}(\mathrm{OPT}_k)}\right)$ then:

$$f(\mathrm{GREEDY}_r) \geq (1 - \varepsilon)\, f(\mathrm{OPT}_k)$$

And this is tight up to a constant factor.

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- We expect vectors in OPT_k to be well-conditioned (think "almost orthogonal") $\Rightarrow$ $\frac{1}{\sigma_{\min}(\mathrm{OPT}_k)}$ small
  - If $\frac{1}{\sigma_{\min}(\mathrm{OPT}_k)}$ bounded by a constant, then only need $r = O\left(\frac{k}{\varepsilon}\right)$ columns
- Significant improvement upon current bounds: depend on ***worst*** singular value of ***any*** k columns

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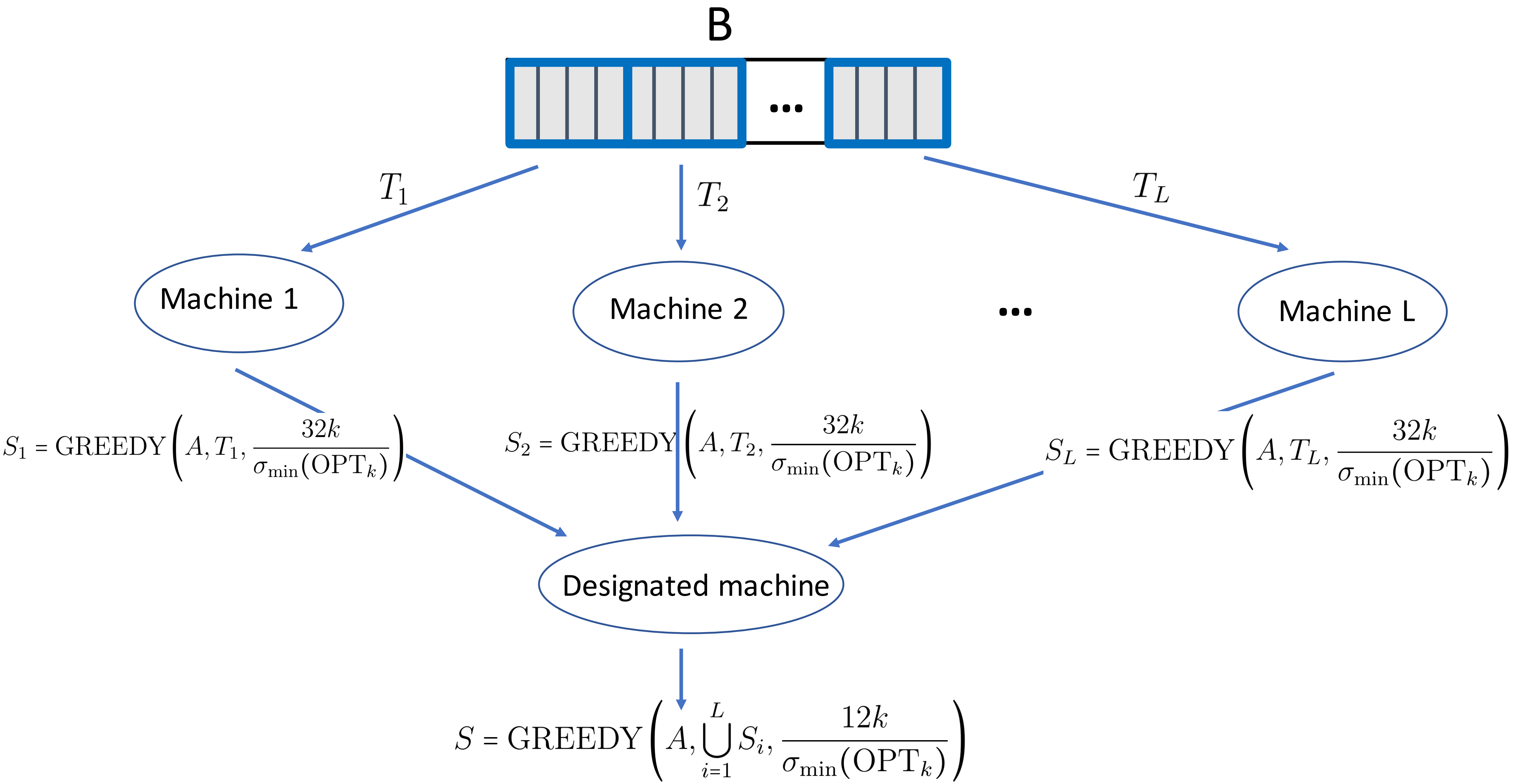
DISTGREEDY: GCSS(A,B,k) with L machines



DISTGREEDY: first observations

- Easy/natural to implement in **MapReduce**

- **2-pass streaming algorithm** in **random arrival** model for columns

- Can also do multiple rounds/epochs. Good for:
  - Massive datasets
  - Getting better approximations (next slide)

Our results: Analysis of DISTGREEDY

Consider an instance GCSS(A, B, k)

1 round result: DISTGREEDY with $r = O\left(\frac{k}{\sigma_{\min}(OPT)}\right)$ gives objective value $\Omega\left(\frac{f(OPT_k)}{\kappa(OPT_k)}\right)$

Condition number $\frac{\sigma_{\max}(OPT_k)}{\sigma_{\min}(OPT_k)}$

Multi-round result: $O(\frac{\kappa(OPT)}{\varepsilon})$ rounds gives objective value $\Omega\Big((1-\varepsilon)f(OPT_k)\Big)$

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Scalable Implementation: GREEDY++

4 optimizations that preserve our $1-\varepsilon$ approximation for $\mathrm{GREEDY}(A \in \mathbb{R}^{m\times n_A},\ B \in \mathbb{R}^{m\times n_B},\ k)$

1. JL Lemma [Johnson & Lindenstrauss 1982, Sarlos 2006]: randomly project to $\boxed{m' \approx \frac{k\log\big(\max(n_A, n_B)\big)}{\varepsilon^2}}$ rows while still preserving k-linear combos $\|\sum_{i=1}^{k} c_i v_i\|_2^2 \in [1\pm\epsilon] \cdot \|\sum_{i=1}^{k} c_i T(v_i)\|_2^2$

2. Projection-Cost Preserving Sketches [Cohen et al. 2015]: sketch A with $\boxed{n'_A \approx \frac{k}{\varepsilon^2}}$ columns.

3. "Stochastic Greedy" [Mirzasoleiman et al. 2015]: each iteration only uses $\boxed{\frac{n_B}{k}\log\frac{1}{\varepsilon}}$ marginal utility calls instead of n_B.

4. Updating A every iteration [Farahat et al. 2013]: after each iteration, remove projections of A and B onto selected column. Reduces complexity of marginal utility from $\boxed{kmn_A \longrightarrow mn_A + mn_B}$

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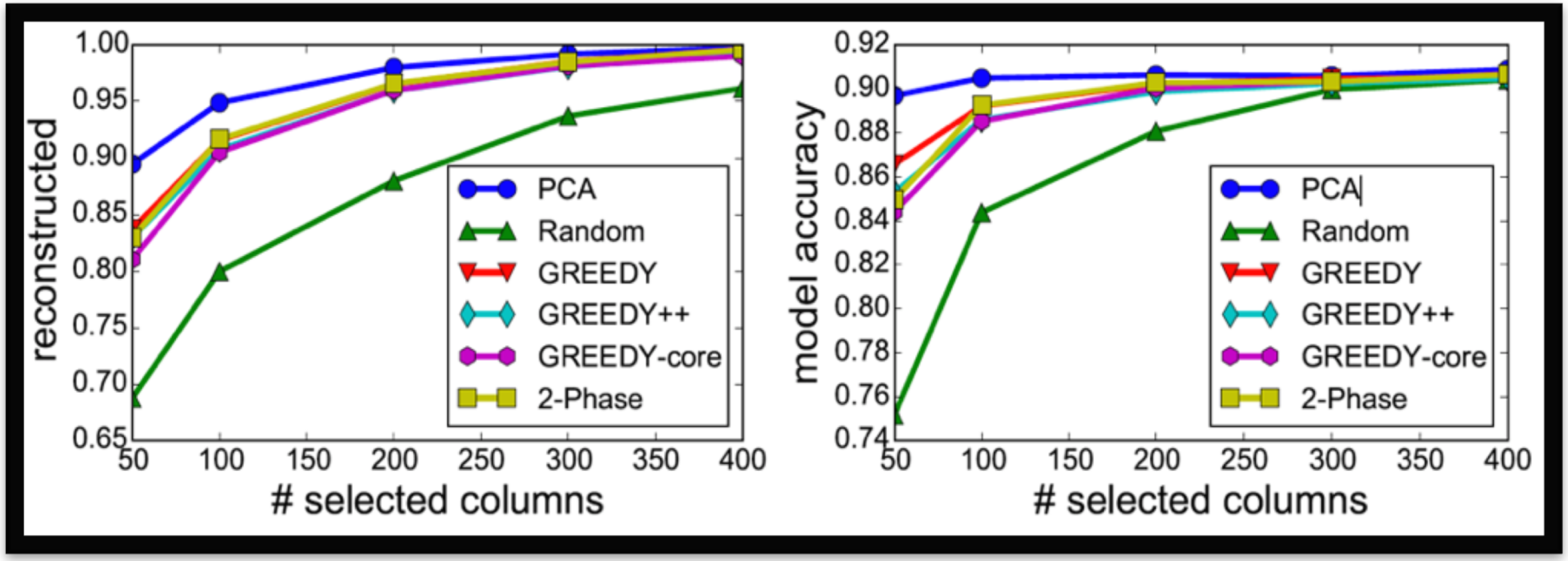
“Small” dataset (mnist): to show **accuracy**

m = **60K instances**

n = **784 features**

10 classes



fraction of matrix covered by selected features

accuracy of LIBLINEAR SVM using selected features

- **Takeaway:** GREEDY, GREEDY++, and GREEDY-core have roughly same accuracy as state-of-the-art

Large dataset (news20.binary) to show **scalability**

m = **15K instances**

n = **100K features**

0.033% nonzero entries

2 classes

k	Rand	2-Phase	DISTGREEDY	PCA
500	54.9	81.8 (1.0)	80.2 (72.3)	85.8 (1.3)
1000	59.2	84.4 (1.0)	82.9 (16.4)	88.6 (1.4)
2500	67.6	87.9 (1.0)	85.5 (2.4)	90.6 (1.7)

classification accuracy using selected features
(Speedup over 2-phase algorithm in parentheses)

- **Takeaway**: DISTGREEDY able to scale to massive datasets while still selecting effective features

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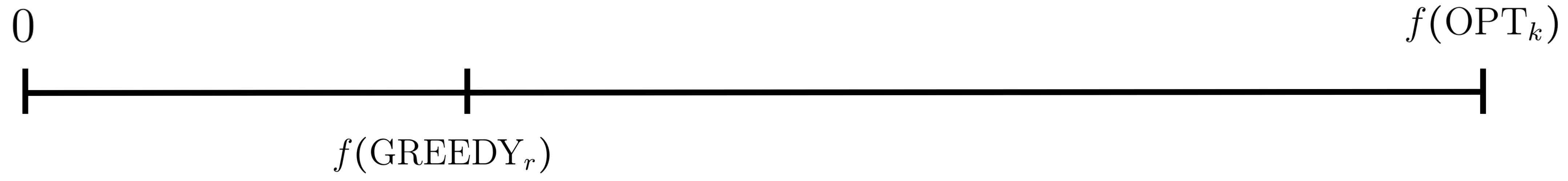
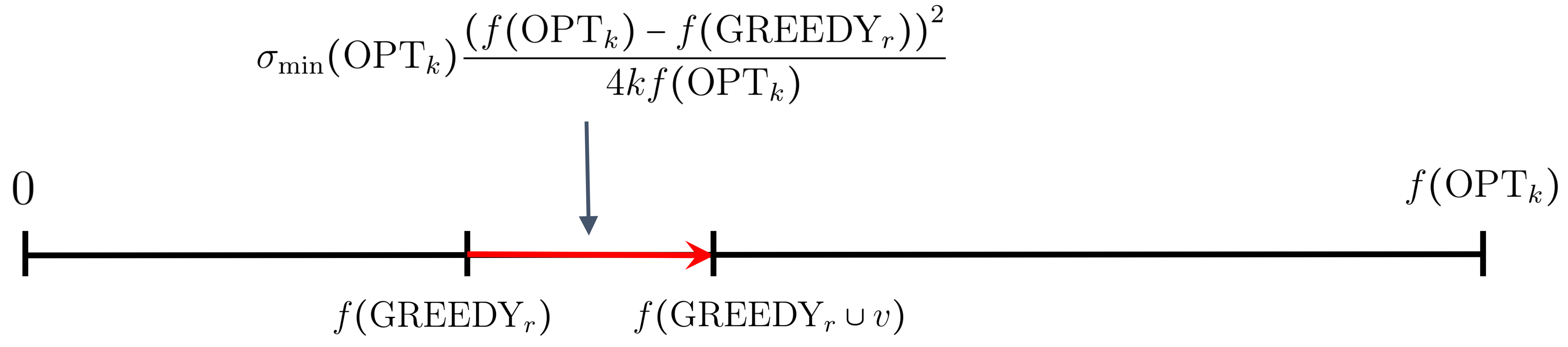


- **Key lemma**: Exists element of OPT_k that gives large marginal gain to GREEDY_r
    - Closes gap to $f(\mathrm{OPT}_k)$
- Similar to submodular functions

Proof sketch: Analysis of GREEDY



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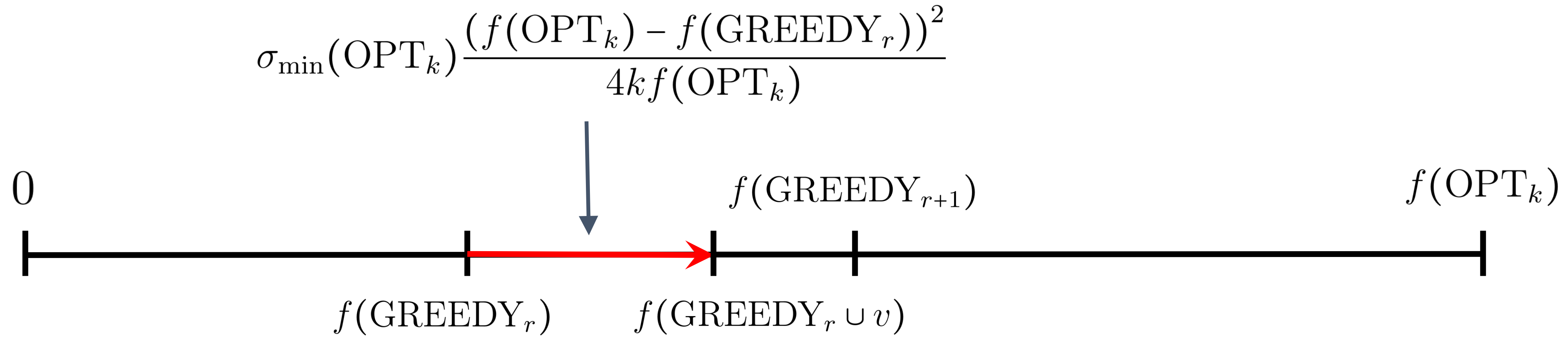

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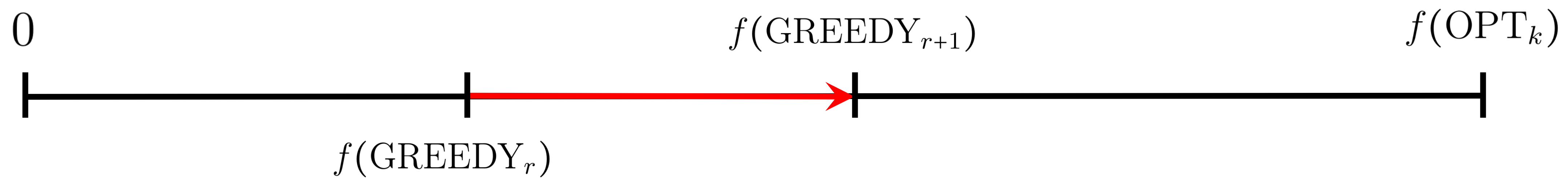

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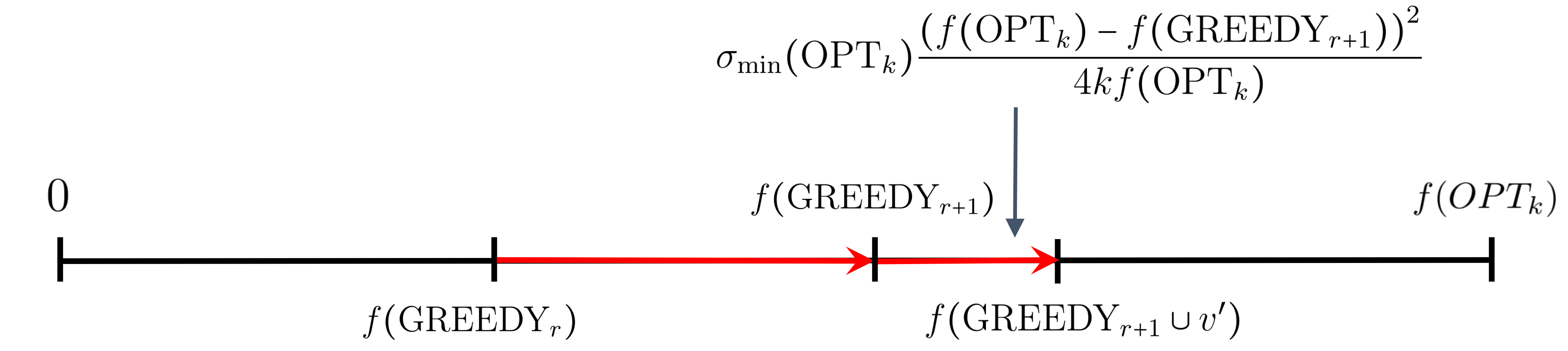

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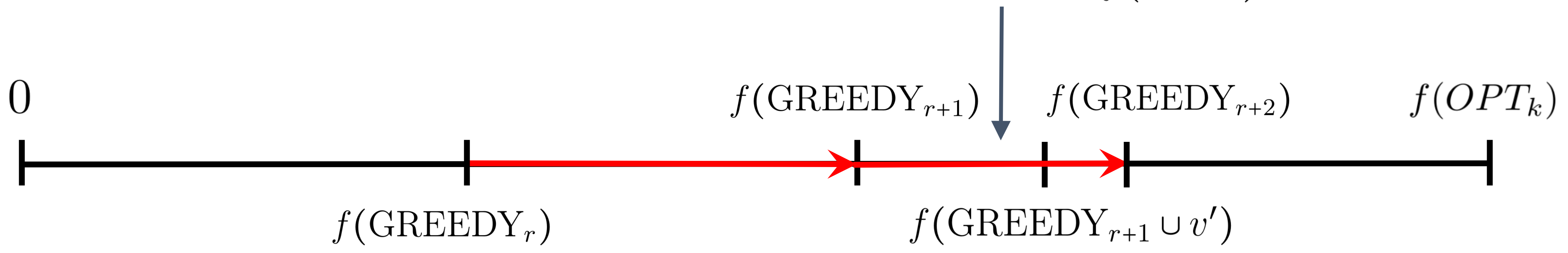
Proof sketch: Analysis of GREEDY

$$\sigma_{\min}(\mathrm{OPT}_k)\frac{(f(\mathrm{OPT}_k) - f(\mathrm{GREEDY}_{r+1}))^2}{4kf(\mathrm{OPT}_k)}$$



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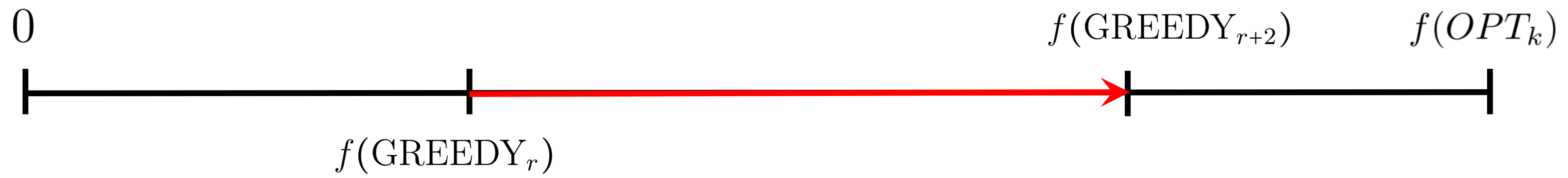

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Questions?