Optimal Resource Allocation to Control Epidemic Outbreaks in Networked Populations

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Presentation Outline

- 0. Motivation: Epidemic processes
- 1. Dynamic Modeling: Epidemics in networks
- 2. Spreading Control Criterion: Exponential die-out
- 3. Optimal Resource Allocation: Geometric programming
- 4. Numerical Simulations: Air transportation network
- 5. Extensions: Generalized models, data-driven allocation

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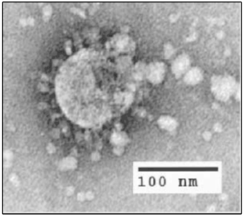
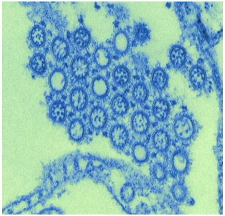
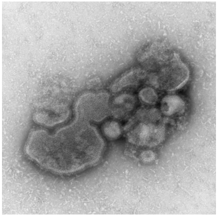
Motivation: Recursive Pandemics



SARS (2002-2003)

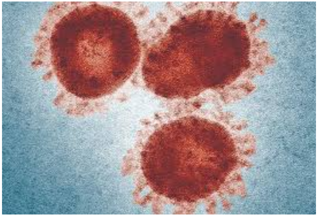


Swine flu (2009-2010)



Avian flu (2013)

Ebola (2014-2016)



COVID19 (2019-?)

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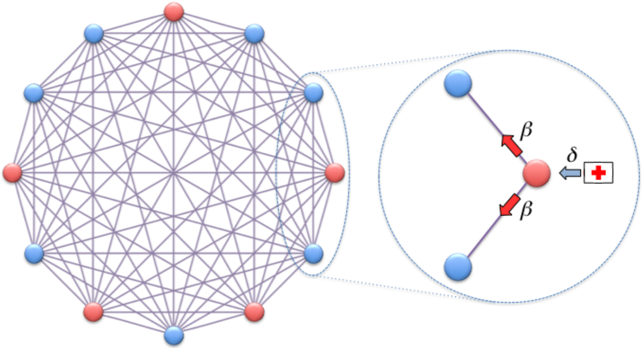
How to model epidemic processes?

Classical models are based on the following assumptions:

- Population classified into compartments (healthy, sick, quarantine...)
- Homogeneous mixing of individuals in an unstructured population

Example: Classical Susceptible-Infected-Susceptible (SIS) model:

- Two compartments: Susceptible (**S**) and Infected (**I**)
- Two parameters: Spreading rate β and recovery rate δ.



Epidemic Dynamics in 'Mixed' Populations

Classical SIS dynamics: Two dynamic variables $p_S(t)$ and $p_I(t)$

- $p_S(t)$ account for the fraction of 'healthy' people
- $p_I(t)$ for the fraction of 'sick' people

Deterministic population dynamics:

$$\frac{dp_I}{dt} = \beta p_S p_I - \delta p_I = \frac{-{p_I}^2 + \left(\frac{\delta}{\beta} - 1\right) p_I}{\beta}, \qquad (1)$$

- Notice that $p_S(t) = 1 - p_I(t)$
- Given an initial condition $p_I(0) \in (0, 1)$, the ODE can be solved

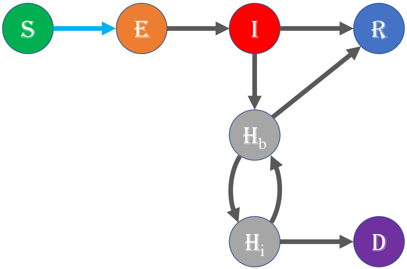
Control criterion: The limit of $p_I(t)$ for $t \to \infty$ presents a bifurcation:

- For $\beta > \delta$, $p_I(t) \to \min\{1 - \delta/\beta, 1\} > 0$ (the disease 'survives')
- For $\beta < \delta$, $p_I(t) \to 0$ (the disease is '**eradicated!**')

Additional compartments can be used to add realism to the dynamics[1]:

- **E**: Exposed individuals with a latent period (i.e., incubating)
- **R**: Recovered individuals with permanent immunity
- **D**: Deceased individuals
- **H**: Hospitalized in bed or in ICU

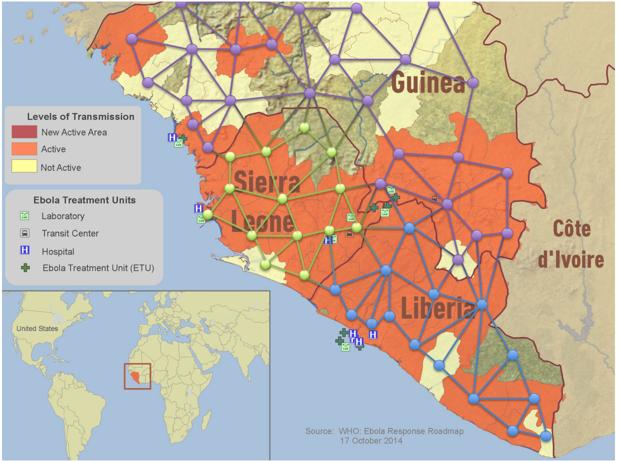
More complex transition diagrams: COVID-19



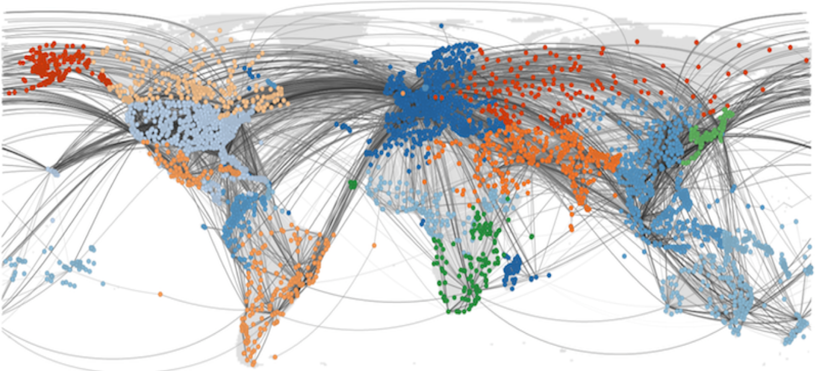
[0] See "*Analysis and control of epidemics: A survey of epidemic processes on complex networks*" IEEE CSM 2016, for a thorough review.

How about 'Networked' Populations?

Network of districts affected by the 2014 Ebola Outbreak:



An even more complex network for COVID-19:



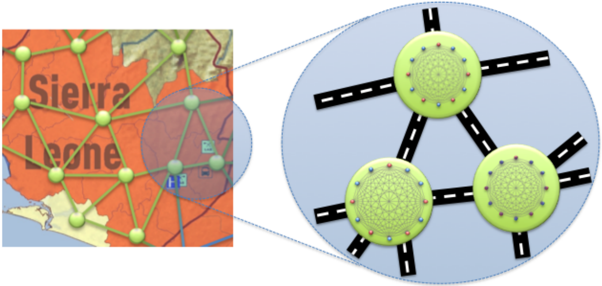
‘Networked’ Metapopulation Models

Network of districts:

- Nodes in the network are *subpopulations* (e.g., districts, towns)
- Links represent *traffic* between subpopulations (e.g., air, road, train)

Networked dynamic model:

- For each subpopulation, there is an internal compartmental dynamics
- These dynamics are coupled through the edges of the network



Consider the 'networked' SIS metapopulation dynamics:

- $p_i^S(t)$ and $p_i^I(t) = p_i(t) = 1 - p_i^S(t)$ denote the fractions of healthy and infected people in the subpopulation of node i at time $t \geq 0$
- $\delta_i > 0$ accounts for the average curing rate of node i
- $\alpha_i > 0$ accounts for the internal infection rate of node i
- $\beta_{ij} > 0$ accounts for the rate of spreading from node j to node i

Disease evolution in a networked population: For each $i = 1, \ldots, N$

$$\frac{dp_i}{dt} = \overbrace{\alpha_i p_i^S p_i}^{\text{Intra-node infection}} - \overbrace{\delta_i p_i}^{\text{Internal recovery}} + \overbrace{\sum_{j \neq i} \beta_{ij} p_i^S p_j}^{\text{Inter-node infection}}$$

substituting $p_i^S = 1 - p_i$ and rearranging terms we obtain:

$$\frac{dp_i}{dt} = \overbrace{\alpha_i p_i - \delta_i p_i + \sum_{j \neq i} \beta_{ij} p_j}^{\text{Linear terms}} - \overbrace{\alpha_i {p_i}^2 - \sum_{j \neq i} \beta_{ij} p_i p_j}^{\text{Nonpositive nonlinear terms (N.N.T.)}}$$

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Question: Under what conditions is the disease 'eradicated'? How fast?

The population dynamics can be written in matrix-vector form:

$$\frac{d}{dt}\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1 - \delta_1 & \beta_{12} & \dots & \beta_{1N} \\ \beta_{21} & \alpha_2 - \delta_2 & \dots & \beta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} & \beta_{N2} & \dots & \alpha_N - \delta_N \end{pmatrix}}_{A-D+B} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} - N.N.T.$$

where $A = diag(\alpha_i)$, $B = [\beta_{ij}]$, and $D = diag(\delta_i)$.

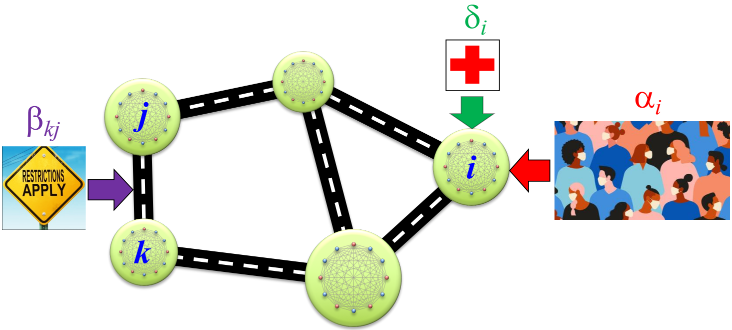
Theorem (Global exponential stability[1])

For any initial condition $\mathbf{p}(0) \in (0,1)^N$, the infection vector $\mathbf{p}(t)$ will decay at an exponential decay rate $\varepsilon > 0$, if and only if,

$$\Re\{\lambda_i(A - D + B)\} \leq -\varepsilon \text{ for all } i = 1, \dots, N.$$

[1]See "*Optimal resource allocation for network protection against spreading processes*," IEEE TCNS, 2014.

Setup: Consider a networked SIS metapopulation model



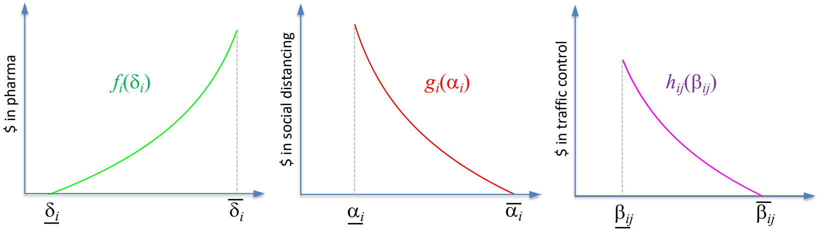
Available Resources: Control an epidemic outbreak using

- Pharma resources able to increase recovery rates δ_i
- Social distancing able to decrease internal infection rates α_i
- Traffic control able to reduce spreading rates β_{ij}, where $\beta_{ij} \neq \beta_{ji}$

Network Protection Problem

We can invest in these resources, assuming costs:

- Pharma cost of tuning the recovery rates $f_i(\delta_i)$
- Social distancing cost to decrease the internal infection rates $g_i(\alpha_i)$
- Traffic control cost aiming to reduce spreading rates $h_{ij}(\beta_{ij})$



Problem: Find the cost-optimal allocation of epidemic-control resources to eradicate any potential outbreak at the *fastest decay rate*[2] possible

[2] In DT models, the decay rate becomes the *effective reproductive rate, R_t*.

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Some Notation

A few **definitions**:

- Consider a weighted, directed graph $\mathcal{G}$ with N nodes and a (nonnegative and asymmetric) adjacency matrix $W = [w_{ij}]$
- $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{C}$ are the eigenvalues of W
- The spectral radius of W is defined as $\rho = \max\{|\lambda_i|\}_{i=1}^N$
- A graph is strongly connected (S.C.) if there exists a directed path between every pair of vertices

Perron-Frobenius lemma: If $\mathcal{G}$ is S.C., then

- $\rho > 0$ is a simple eigenvalue of W
- $W\mathbf{u} = \rho\mathbf{u}$, for some eigenvector $\mathbf{u} > 0$ (component-wise)
- $\rho = \inf\{\lambda > 0 \colon W\mathbf{u} < \lambda\mathbf{u},\ \mathbf{u} > 0\}$

Geometric Programming

Geometric programs are quasiconvex optimization problems that can be convexify via a logarithmic change of variables

- Let $x_1, \dots, x_n > 0$ denote n positive decision variables
- Define $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$
- In the context of GP, a *monomial* $m(\mathbf{x})$ is defined as $m(\mathbf{x}) = c x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ with $c > 0$ and $a_i \in \mathbb{R}$
- A *posynomial* function $q(\mathbf{x})$ is defined as a sum of monomials, i.e., $q(\mathbf{x}) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}$, where $c_k > 0$.

A Geometric Program[3] is an optimization problem of the form:

$$\begin{aligned} \text{minimize } & p(\mathbf{x}) \\ \text{subject to } & q_i(\mathbf{x}) \leq 1,\ i = 1, ..., m, \\ & m_i(\mathbf{x}) = 1,\ i = 1, ..., p, \end{aligned} \tag{2}$$

where q_i and p are posynomial functions, m_i are monomials

[3] For more information about GP's, see Boyd's monograph "A Tutorial on Geometric Programming".

Problem statement:

Given the following elements:

- A directed network of traffic links connecting subpopulations
- Cost functions $f_i(\delta_i)$, $g_i(\alpha_i)$, and $h_{ij}(\beta_{ij})$
- Box constraints $\underline{\delta}_i \leq \delta_i \leq \overline{\delta}_i$, $\underline{\alpha}_i \leq \alpha_i \leq \overline{\alpha}_i$, and $\underline{\beta}_{ij} \leq \beta_{ij} \leq \overline{\beta}_{ij}$
- A total **budget C** to be invested in containment resources[4]

Find the cost-optimal joint allocation of epidemic-control resources to *maximize the exponential decay rate* ε of any outbreak

[4]Alternatively, we could impose a constraint on the a desired decay rate, instead of a budget C. For more details, see "*Optimal resource allocation for network protection against spreading processes*" IEEE TCNS, 2014.

Budget-Constrained Network Protection

Mathematical formulation: $A = diag(\alpha_i)$, $B = [\beta_{ij}]$, and $D = diag(\delta_i)$

$$\underset{\varepsilon, A, B, D}{\text{maximize}} \ \varepsilon \tag{3}$$

$$\text{subject to } \Re\{\lambda_i(A - D + B)\} \leq -\varepsilon \text{ for all } i, \tag{4}$$

$$\sum_i g_i(\alpha_i) + \sum_i f_i(\delta_i) + \sum_{i,j} h_{ij}(\beta_{ij}) \leq \mathbf{C}, \tag{5}$$

$$\underline{\alpha}_i \leq \alpha_i \leq \overline{\alpha}_i \text{ and } \underline{\delta}_i \leq \delta_i \leq \overline{\delta}_i \text{ for all } i, \tag{6}$$

$$\underline{\beta}_{ij} \leq \beta_{ij} \leq \overline{\beta}_{ij} \text{ for all } (i,j), \tag{7}$$

In what follows, we solve this allocation problem for *directed* networks using geometric programming[5]

For clarity in our exposition, we will only consider non-pharma actions (i.e., the recovery rates, δ_i, are out of our control)

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[5] For undirected networks and cost functions satisfying a mild convexity constraint, the problem can be solved using SDP; see "*Optimal Vaccine Allocation to Control Epidemic Outbreaks in Arbitrary Networks*" IEEE CDC, 2013.

Sketch of our Approach (informal[6])

The main difficulty stems from the spectral condition (3) in the optimization program. We transform Condition (3) by following these steps:

1. Define $\boldsymbol{\Delta} = \max\{\delta_i : i = 1, \ldots, N\}$; hence,

$$\Re\{\lambda_i(A - D + B)\} \leq -\varepsilon \iff \Re\{\lambda_i(A + \underbrace{\boldsymbol{\Delta}\mathbf{I}_N - D}_{D^c} + B)\} \leq \underbrace{\boldsymbol{\Delta} - \varepsilon}_{\varepsilon^c}. \quad (8)$$

2. We have that $A + D^c + B \geq 0$; hence, according to Perron-Frobenius

$$(8) \text{ for all } i \iff \rho(A + D^c + B) \leq \varepsilon^c. \quad (9)$$

3. Using the variational result in the P.F. lemma, we obtain

$$(9) \iff \inf\{\lambda > 0 : (A + D^c + B)\mathbf{u} < \lambda\mathbf{u}, \mathbf{u} > 0\} \leq \varepsilon^c.$$

4. The vector inequality $(A + D^c + B)\mathbf{u} < \lambda\mathbf{u}$ can be written, component-wise, as

$$\alpha_i u_i + \delta_i^c u_i + \sum_{j=1}^N \beta_{ij} u_j < \lambda u_i \text{ for all } i = 1, \ldots, N$$

which is equivalent to the following set of posynomial inequalities:

$$\lambda^{-1}\alpha_i + \lambda^{-1}\delta_i^c + \lambda^{-1}\sum_{j=1}^N \beta_{ij} u_i^{-1} u_j < 1 \text{ for all } i = 1, \ldots, N$$

[5] More details in "*Optimal resource allocation for network protection against spreading processes*," IEEE TCNS, 2014.

Main Result: Budget-constrained allocation (non-pharma actions)

Assuming that g_i and h_{ij} are *posynomial cost functions*[7], solve the following Geometric Program:

$$\begin{aligned}
\underset{\lambda,\{u_i,\alpha_i\}_i,\{\beta_{ij}\}_{i,j}}{\text{minimize}} \quad & \lambda \\
\text{subject to} \quad & \lambda^{-1}\alpha_i + \lambda^{-1}\delta_i^c + \lambda^{-1}\textstyle\sum_j \beta_{ij} u_i^{-1} u_j \leq 1 \text{ for all } i, \\
& \textstyle\sum_i g_i(\alpha_i) + \sum_{i,j} h_{ij}(\beta_{ij}) \leq \mathbf{C}, \\
& \beta_{ij} \leq \overline{\beta}_{ij} \text{ and } \beta_{ij}^{-1} \leq \underline{\beta}_{ij} \text{ for all } (i,j), \\
& \alpha_i \leq \overline{\alpha}_i \text{ and } \alpha_i^{-1} \leq \underline{\alpha}_i \text{ for all } i.
\end{aligned}$$

The optimal decay rate of the epidemics with a budget **C** is given by $\varepsilon^\star = \Delta - \lambda^\star$.

[7] More generally, g_i and h_{ij} have to be convex in log-log scale, i.e., $G_i(\mathbf{y}) = \log(g_i(\exp(\mathbf{y})))$ and $H_{ij}(\mathbf{y}) = \log(h_{ij}(\exp(\mathbf{y})))$ are convex functions of $\mathbf{y}$.

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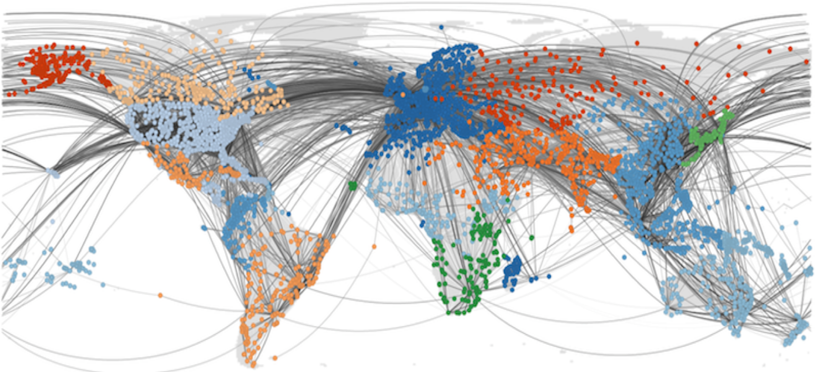
Numerical Illustration: Air Transportation Network

Find the cost-optimal non-pharmaceutical strategy against an SIS-type pandemic propagating through the *air transportation network*:

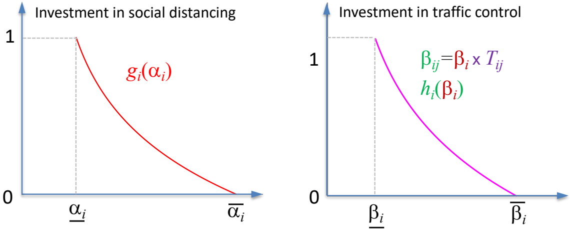


[7] Disclaimer: These following simulations were run *before* the current COVID-19 outbreak and were conceived as an academic illustration in 2014. The optimal distribution of resources shown below does not apply to COVID-19.

Numerical Illustration: Setup

Setup: We consider the following elements

- World-wide air traffic network: airports (nodes) and traffic (weighted links)
- Fixed recovery rates δ_i (non-pharma actions only)
- Cost of *social distancing* $g_i(\alpha_i)$ and *traffic control* $h_{ij}(\beta_{ij})$ (shown below)
- A total budget **C**



Objective: Find the cost-optimal allocation of non-pharma resources to maximize the exponential decay rate of a disease with a given budget

Investments on traffic control vs social distancing over airports:

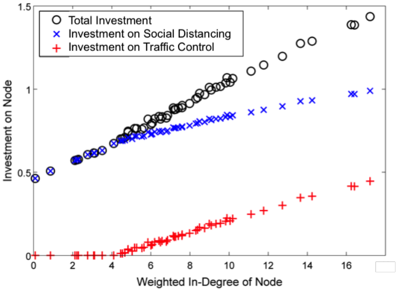


Figure: (*Left*) a scatter plot with the investment for each airport vs. the incoming traffic (in MPPY), and (*right*) a scatter plot with the investment versus PageRank centralities of the airport.

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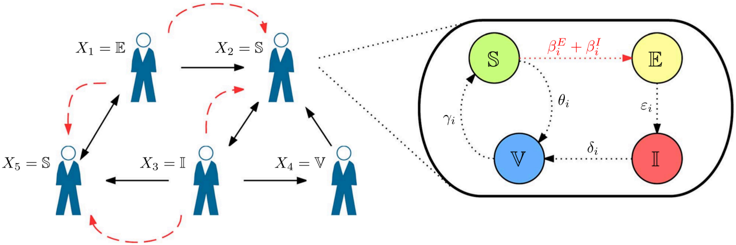
Extension 1: Generalized epidemic models

We can analyze epidemic models with a variety of intermediate states:

- **Exposed** to the disease (i.e., incubating without symptoms),
- **Vigilant** about the spread (i.e., informed and careful), etc.

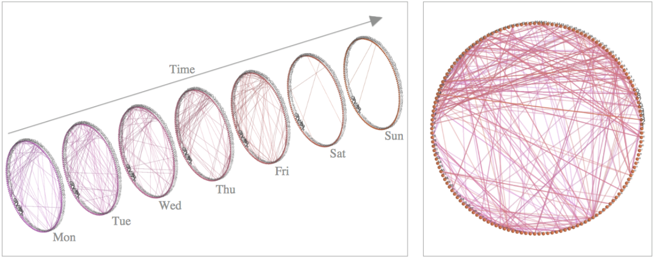


An optimal allocation of resources, including *educational campaigns*, can be found via Geometric Programming in the SEIV model[8].

[8] See "*Optimal resource allocation for control of networked epidemic models*," IEEE TCNS, 2016.

Extension 2: Time-Switching Networks

In real life, epidemic processes take place in time-varying networks:



Temporal Network (24-h window) and aggregated static graph[9].

- Aggregrated static graphs induce strong biases in the epidemics.
- In [Ogura and P., 2015a][10], we develop a framework to analyze and control epidemic processes in time-varying networks.

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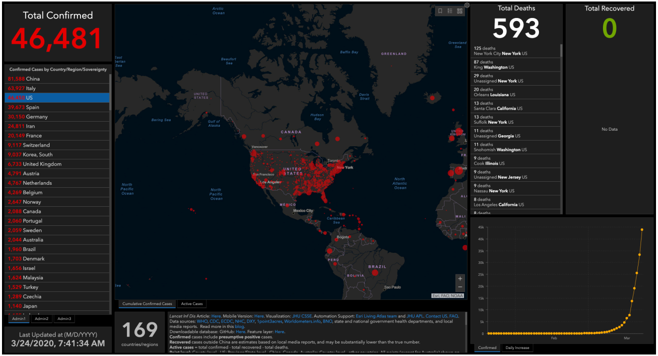
[10]Figure from Perra et al., 2012.
[10]See "*Stability of spreading processes over time-varying large-scale networks*," IEEE TNSE, 2015.

In practice, the spreading and recovery rates of the systems are not explicitly given. Instead, we usually have access to (unreliable) data[11]:



- In [Hayhoe et al., 2023] we developed a metalearning approach and in [Han et al., 2015] a data-driven formulation based on conic GP.

[11] Figure and data from CSSE@jhu

Questions? A few references below...

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