(Near)-substitute Preferences and Equilibria with Indivisibilities

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What?

Competitive equilibria (CE) with indivisible goods.

1. New sufficient condition for existence of CE that supersedes all prior ones (with two exceptions).
2. Identify prices at which the excess demand for each good is bounded by a preference parameter *independent* of the size of the economy (social approximate equilibrium).

Why?

CE outcomes are a benchmark for the design of markets for allocating goods and services.

When they exist they are in the core and can be pareto optimal.

Under certain conditions they satisfy fairness properties like equal treatment of equals and envy-freeness.

Prior Work

Restrict preferences

Kelso & Crawford (1982), Gul & Stachetti (1999), Danilov, Koshevoy & Murota (2001), Sun & Yang (2006), Baldwin & Co. (2020)

Drown indivisibilities in 'large' markets.

Azevedo, Weyl & White (2013)

Prior Work

Eliminate the indivisibilities via lotteries.

Hylland & Zeckhauser (1979), Budish et. al. (2013), Gul, Pesendorfer & Zhang (2020)

Approximate CE outcomes based on cardinal notions of welfare; approximations scale slowly with size of economy.

Dobzinski et al (2014), Feldman et al (2014)

Social Approximate Equilibrium; mismatch between supply and demand diminishes with size of economy.

Broome (1971), Dierker (1970), Starr (1969)

Notation

$M =$ set of indivisible goods (courses).

Bundle of goods is denoted by vector $x \in \{0,1\}^m$ (**Single Copy Demand**)

Price vector denoted $p \in \mathbb{R}^m$, cost of bundle x is $p \cdot x$.

X a finite set of bundles one can choose from.

Utility for a bundle x when endowed with b units of money is denoted $U(x, b - p \cdot x)$.

Conditions

1. $U(\vec{0}, 0) = 0$.
2. $U(x, b - p \cdot x) = -\infty$ for $x \notin X$.
3. $U(x, t)$ is continuous and strictly decreases with t for $x \in X$.
4. There exists $B > 0$ such that if $p \cdot x \geq B$, then, $U(x, b - p \cdot x) < U_j(\vec{0}, b)$.

Quasi-linearity: $U(x, b - p \cdot x) = v(x) + b - p \cdot x$.

Notation

Choice correspondence, denoted $Ch(p)$:

$$Ch(p) = \arg\max\{U(x, b - p \cdot x) : x \in X\}.$$

$(x - y)^+$ is vector whose i^{th} component is $\max\{x_i - y_i, 0\}$.

Equilibrium

N set of agents, $s_i \in \mathbb{Z}_+$ the supply of good $i \in M$.

$\vec{s} \in \mathbb{Z}^m_+$ is the supply vector.

$\vec{b}$ the vector of cash endowments.

An economy is the collection $\{\{U_j\}_{j \in N}, \vec{b}, \vec{s}\}$.

Equilibrium

A CE for the economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ is a price vector p and demands $x^j \in Ch_j(p)$ for all $j \in N$ such that $\sum_{j\in N} x^j \leq s$ with equality for each $i \in M$ for which $p_i \neq 0$.

A α-approximate CE for the economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ is a competitive equilibrium for the economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s'}\}$, where $|s_i - s_i'| \leq \alpha$ for every good $i \in M$.

Δ-substitutes

A utility function $U(x, b - p \cdot x)$ satisfies Δ-substitutes if for any price vector p and endowment b, either $|Ch(p)| = 1$ or there exist two different bundles $x, y \in Ch(p)$ such that

$||(x-y)^+||_1 \leq \Delta$ and $||(y-x)^+||_1 \leq \Delta$.

THEOREM: If all agent's preferences satisfy Δ-substitutes, then, for every supply vector s the economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ has a $2(\Delta - 1)$-approximate CE.

Prior Sufficient Conditions

Gross Substitutes $\subset$ 1-Substitutes.

Single improvement $\subset$ 1-Substitutes.

No complementarities $\subset$ 1-Substitutes.

Net substitutes $\subset$ 1-Substitutes.

Pseudo-equilibrium

A price vector p and $x^j \in conv(Ch_j(p))$ for all $j \in N$ is called a pseudo-equilibrium if $\sum_{j \in N} x^j \leq s$ with equality for every good $i \in M$ with $p_i \neq 0$.

Rounding Lemma

Polytope Q is **binary** if all of its extreme points are 0-1 vectors and denote its set of extreme points by $ext(Q)$.

A binary polytope is Δ-**uniform** if each of its edges has at most Δ positive and at most Δ negative coordinates.

Rounding Lemma

Let $Q_1, \ldots, Q_n$ be binary Δ-uniform polytopes in $\mathbb{R}^m$.

Each $y = (y^1, \ldots, y^n) \in Q_1 \times \ldots \times Q_n$ can be expressed as a convex combination of points in $ext(Q_1) \times \ldots \times ext(Q_n)$.

For each $(z^1, \ldots, z^n) \in ext(Q_1) \times \ldots \times ext(Q_n)$ in the support of the convex combination,

$$||\sum_{j=1}^{n} z^j - \sum_{j=1}^{n} y^j||_\infty < 2\Delta - 1.$$

Shapley-Folkman-Starr # 1

Let $P_1, \ldots, P_n$ be binary polytopes in $\mathbb{R}^m$ with $n > m$.

For any $y \in conv(P_1) + \ldots + conv(P_n)$ there exist

$z^j \in conv(P_j)$ for all j of which at least $n - m$ are in $ext(P_j)$ such that

$$||y - \sum_{j=1}^{n} z^j||_\infty \leq m.$$

Shapley-Folkman-Starr # 2

Let $P_1, \ldots, P_n$ be polytopes in with $n > m$.

The diameter of each P_i is no larger than $d < \infty$.

For any $y \in conv(P_1) + \ldots + conv(P_n)$ there exist

$z^j \in conv(P_j)$ for all j of which at least $n - m$ are in $ext(P_j)$ and

$$||y - \sum_{j=1}^{n} z^j||_2 \leq \frac{d\sqrt{m}}{2}.$$

Course Allocation

Budish (2011): Social approximate equilibria with approximately equal incomes (A-CEEI).

In expectation, each agent receives the same endowment of artificial currency.

Compute a social approximate CE.

Euclidean distance between supply and demand vector is at most $m\sqrt{\Delta}$.

Course Allocation

Ordinal preference of student j is $\succeq_j$.

$v_j(x) \geq 0$ utility function that represents these preferences.

Given a budget of \$1, and price p, the **auxiliary** utility of agent j for bundle x is

$$U_j^{\epsilon}(x,p) := v_j(x) + \min\{0, \log(\frac{1 - p \cdot x}{\epsilon})\}$$

Course Allocation

$Ch_j^{\epsilon}(p)$ denote the choice correspondence of auxiliary utility function.

Let $x \in Ch_j^{\epsilon}(p)$, then, there exists a budget b such that $1 \leq b < 1 + \epsilon$ and

$$x = max_{(\succeq_j)}\{x' \in X_j \text{ and } p \cdot x' \leq b\}.$$